Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS SPECIALIST

Calculator-assumed

ATAR course examination 2017

Marking Key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2017/69888

Section Two: Calculator-assumed **65% (97 Marks)**

Question 9 **(6 marks)**

The time T in minutes that a particular flight arrives later than its scheduled time is uniformly distributed with $-30 \le T \le 60$. The population mean is $\mu(T) = 15$ and the population variance is $\sigma^2(T) = 675$.

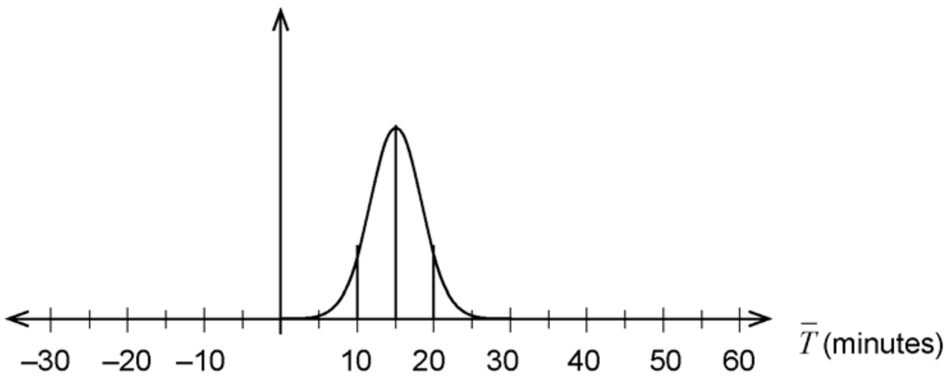
A sample of 30 arrival times is taken and the sample mean $\overline{T}$ is calculated.

(a) Determine $P\left(10 \le \overline{T} \le 20\right)$ correct to 2 decimal places. (3 marks)

Solution
The sample mean will be normally distributed. $\overline{T} \sim N\left(15, \dfrac{675}{30}\right)$ i.e. $\sigma^2\left(\overline{T}\right) = 22.5 \qquad \sigma\left(\overline{T}\right) = 4.74 \quad (2\text{ d.p.})$ $P\left(10 \le \overline{T} \le 20\right) = 0.71 \quad (2\text{ d.p.})$
Specific behaviours
✓ states that the sample mean is normally distributed ✓ states the parameters of the sample mean ✓ calculates the correct probability

(b) If a large number of samples, each with 30 arrival times is taken, sketch the likely distribution of the sample mean $\overline{T}$ below.

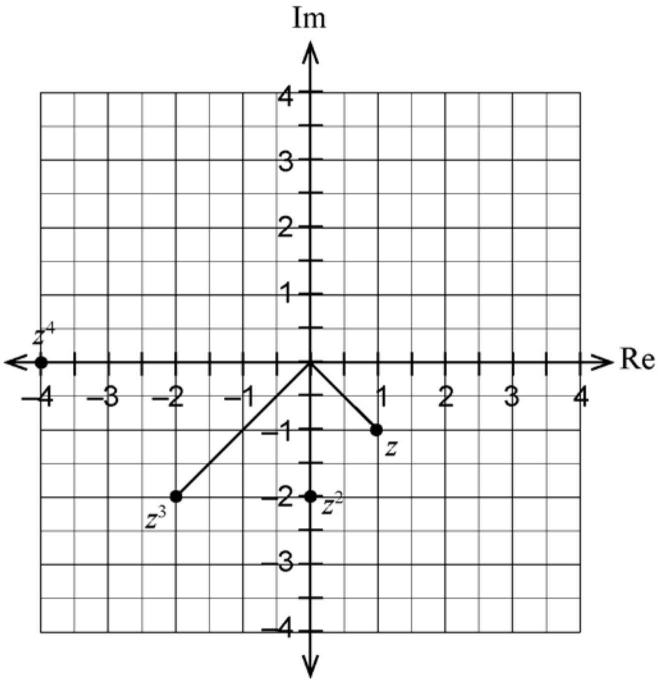
In the diagram indicate or refer to the calculation from part (a). (3 marks)



Solution
As above. There is approx. 71% of the total area under the curve for $10 \le \overline{T} \le 20$.
Specific behaviours
✓ indicates a normal distribution centred at $\mu\left(\overline{T}\right) = 15$ ✓ indicates a standard deviation of approx. 5 minutes ✓ refers to the probability from part (a)

Question 10 **(7 marks)**

Consider $z = 1 - i$ shown in the complex plane below.



(a) Express z in polar form. (1 mark)

Solution
$z = \sqrt{2}cis\left(-\frac{\pi}{4}\right)$
Specific behaviours
✓ states the correct polar form (both modulus and argument)

(b) Hence express z^2, z^3 and z^4 in exact polar form. (2 marks)

Solution
$z^2 = \left(\sqrt{2}cis\left(-\frac{\pi}{4}\right)\right)^2 = 2cis\left(-\frac{\pi}{2}\right)$ $z^3 = \left(\sqrt{2}cis\left(-\frac{\pi}{4}\right)\right)^3 = 2\sqrt{2}cis\left(-\frac{3\pi}{4}\right)$ $z^4 = \left(\sqrt{2}cis\left(-\frac{\pi}{4}\right)\right)^4 = 4cis(-\pi) \quad or \quad 4cis(\pi)$
Specific behaviours
✓ writes the correct modulus for each power   ✓ writes the correct argument for each power

Question 10 (continued)

(c) Sketch the complex numbers z^2, z^3, z^4 as vectors in the given Argand diagram.

(2 marks)

Solution
As shown in the Argand diagram
Specific behaviours
✓ indicates the correct modulus for each vector ✓ indicates the correct argument for each vector

Consider the geometric transformation(s) applied to transform $z \to z^2 \to z^3 \to z^4$ etc.

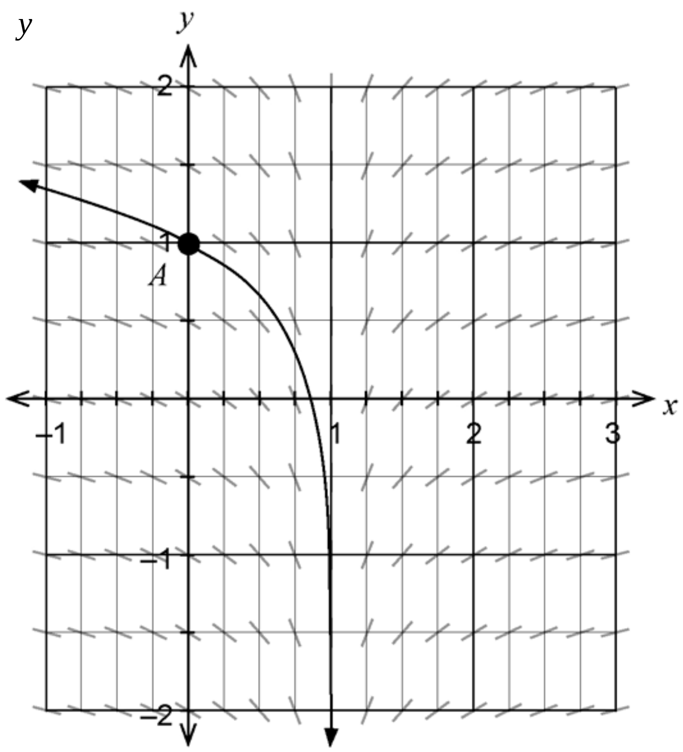
(d) Describe the geometric transformation(s) performed by successive multiplication by z.

(2 marks)

Solution
Successive multilplication by z results in the modulus changing by a factor of $\sqrt{2}$ and the argument decreasing by 45º. Geometric description: Each vector is ENLARGED by a factor of $\sqrt{2}$. Each vector is ROTATED clockwise (about origin) by 45º.
Specific behaviours
✓ describes the change in the modulus as an enlargement by factor $\sqrt{2}$ ✓ describes the change in the argument as a clockwise rotation by 45º

Question 11 **(7 marks)**

A series of magnets is placed under a glass pane and some iron filings are sprinkled onto the glass. The orientation or slope of the iron filings, as determined by the magnetic field, is shown below. One of the lines of magnetic force that passes through the point $A\ (0,1)$ is also shown.



The slope field is given by $\frac{dy}{dx} = \frac{1}{2x-2}$, $x \neq 1$.

(a) Determine the value of the slope field at the point $A\ (0,1)$. (2 marks)

Solution
Substitute $x = 0$ into $\frac{dy}{dx} = \frac{1}{2(0)-2} = -0.5$ i.e. the slope field has a value of -0.5 at point A
Specific behaviours
✓ substitutes $x = 0$ into the expression for $\frac{dy}{dx}$ ✓ evaluates the slope field correctly

Question 11 (continued)

(b) Explain the orientation of the iron filings at $x=1$. (1 mark)

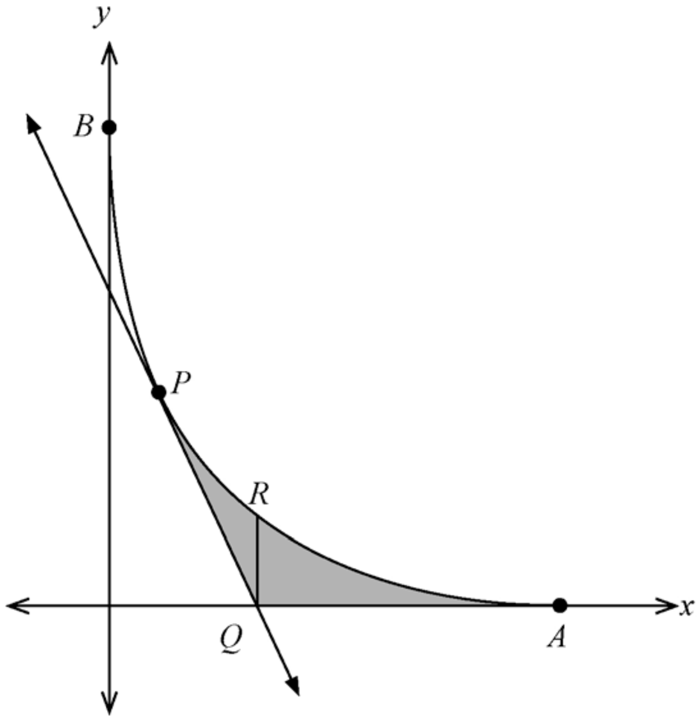
Solution
As $x \to 1$, $\left\lvert \frac{dy}{dx} \right\rvert \to \infty$ hence the orientation becomes infinitely steep i.e. vertical OR that the slope is undefined at $x=1$
Specific behaviours
✓ explains appropriately with reference to the very high slope values as $x \to 1$ OR states that the slope is undefined at $x=1$

(c) Determine the equation for the line of force that passes through the point $A\ (0,1)$.
(4 marks)

Solution
From $\frac{dy}{dx} = \frac{1}{2x-2}$ then $y = \int \frac{1}{2x-2}\,dx = \frac{1}{2}\ln\lvert 2x-2\rvert + c$ Using $(0,1)$ $1 = \frac{1}{2}\ln\lvert 2(0)-2\rvert + c$ $\therefore\ c = 1 - \frac{1}{2}\ln(2)$ i.e. Equation for the line of force is $y = \frac{1}{2}\ln\lvert 2x-2\rvert + \left(1-\frac{1}{2}\ln(2)\right)$ i.e. $y = \frac{1}{2}\ln\lvert 2x-2\rvert + 0.6534...$
Specific behaviours
✓ anti-differentiates using the natural logarithm of an absolute value ✓ anti-differentiates correctly using a factor of one-half ✓ substitutes the coordinates of point A correctly into the anti-derivative function ✓ determines the constant of integration correctly

Question 12 **(8 marks)**

The diagram shows the curve with equation $\sqrt{x}+\sqrt{y}=3$ where points A, B are the intercepts of this curve. A tangent is drawn to the curve at point $P\ (1,4)$.



(a) Show that the equation of the tangent is $2x+y=6$. (3 marks)

Solution
Differentiating implicitly : $\frac{d}{dx}\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right) = \frac{d}{dx}(3)$
$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\left(\frac{dy}{dx}\right) = 0$
$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$
At $(1,4)$ $m = \frac{dy}{dx} = -\frac{\sqrt{4}}{\sqrt{1}} = -2$
Hence equation of tangent: $y-4 = -2(x-1)$
i.e. $y=6-2x$ OR $2x+y=6$
Specific behaviours
✓ differentiates correctly ✓ determines the slope of the tangent correctly ✓ forms the equation of the tangent correctly

Question 12 (continued)

The shaded region is bounded by the curve, the tangent and the x axis.

(b) Determine the exact area of the shaded region. (5 marks)

Solution
y coordinate of the curve at point P is $y = 4$. Considering horizontal rectangles: Area shaded $=$ Region(AQP) $= \int_0^4 \left(3-\sqrt{y}\right)^2 - \left(\frac{6-y}{2}\right) dy$ $= 4$ square units
Specific behaviours
✓ uses the y coordinate for P correctly ✓ forms one integral with the correct limits of integration correctly in terms of y ✓✓ determines the integrand for the integral correctly in terms of y ✓ evaluates the total area correctly

Alternative Solution
x intercept of the tangent at point Q is $(3,0)$. x intercept of the curve at point A is $(9,0)$. Considering vertical rectangles: Area shaded $=$ Region(PRQ) $+$ Region(RQA) $= \int_1^3 \left(3-\sqrt{x}\right)^2 - (6-2x)\,dx + \int_3^9 \left(3-\sqrt{x}\right)^2 dx$ $= 1.2153... + 2.7846...$ $= 4$ square units
Specific behaviours
✓ determines the coordinates for A, Q correctly ✓ forms two integrals with the correct limits of integration in terms of x ✓ determines the integrand for the first integral correctly ✓ determines the integrand for the second integral correctly ✓ evaluates the total area correctly

Question 13 **(13 marks)**

A cable in a bridge is required to support a weight of 10 000 Newtons. Tina tests a random sample of 100 cables from a supplier. The sample mean is found to be 10 300 Newtons and the sample standard deviation 400 Newtons.

(a) Based on Tina's sample, obtain a 95% confidence interval for μ, the population mean cable strength. (4 marks)

Solution
Sample mean is normally distributed $\overline{X} \sim N\left(\mu,\ s_{\overline{X}}{}^{2}\right)$ using $s_{\overline{X}} = \frac{400}{\sqrt{100}} = 40$
For 95% confidence use $k = 1.96 \quad (1.9599..)$
Confidence interval: $10300 - 1.96(40) \ \leq \ \mu \ \leq \ 10300 + 1.96(40)$
i.e. $10221.6 \ \leq \ \mu \ \leq \ 10378.4$ Newtons
Specific behaviours
✓ uses 10 300 as the centre of the interval ✓ calculates the correct standard deviation for the sample mean ✓ uses the correct z score for the 95% confidence level ✓ calculates the upper and lower limits correctly

(b) State whether each of the following statements is true or false. Provide reasons for your answer and state any assumptions.

(i) If another sample of 100 cables is taken, then the sample mean will fall within the confidence interval found at part (a). (2 marks)

Solution
Statement is FALSE. The sample mean is based on another random sample and it is not a certainty that the sample mean will fall within the confidence interval obtained at part (a). i.e. it is not a certainty due to random sampling.
Specific behaviours
✓ states that the statement is false ✓ justifies the answer i.e. it is not a certainty due to random sampling

(ii) If a single cable is selected at random, then the strength of the cable will fall within the confidence interval found at part (a). (2 marks)

Solution
Statement is FALSE. This is a single observation (not a sample mean) of a cable strength. The distribution of a single observation will have a larger variation than the variation of a sample mean and may fall outside of the interval.
Specific behaviours
✓ states that the statement is false ✓ justifies the answer i.e. single observation has a larger variation

Question 13 (continued)

Jon, a colleague of Tina, said, ‘The cable strengths are not normally distributed, so the calculation for the confidence interval is incorrect’.

(c) How should Tina respond to Jon’s comment? (2 marks)

Solution
Tina should inform Jon that he is NOT correct. Not knowing the nature of the underlying distribution of the cable strengths does not make any difference. The sample mean based on a sample size of 100 will be approx. normally distributed irrespective of the population.
Specific behaviours
✓ states that Jon’s statement is not correct ✓ justifies the answer i.e. the sample mean WILL be normally distributed or refers to the large sample size

A different sample of 36 cables is taken and it is found that the standard deviation is 500 Newtons. A confidence interval for the population mean cable strength is determined to be $9900 \le \mu \le 10\,200$.

(d) Determine the confidence level, to the nearest 0.1%, used to calculate this interval. (3 marks)

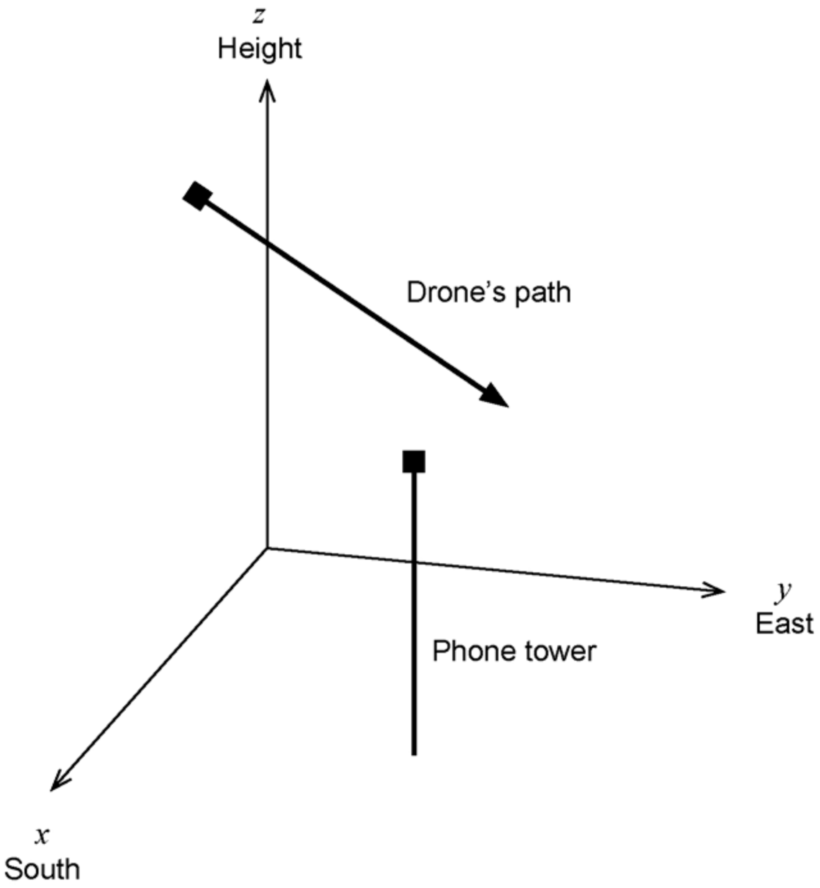
Solution
Given $9900 \le \mu \le 10\,200$ we can infer that the sample mean was $\overline{X} = \dfrac{9900+10200}{2} = 10\,050$ Hence $k \times \dfrac{500}{\sqrt{36}} = 150$ Solving gives $k = 1.8$ $\therefore\ P(-1.8 < z < 1.8) = 0.9281...$ where $z = N(0,1^2)$ standard normal variable Hence the confidence level used was 92.8% (to nearest 0.1%)
Specific behaviours
✓ determines the variation of 150 Newtons either side of the sample mean ✓ solves for the critical z score to yield this variation ✓ determines the confidence level correctly (to nearest 0.1%)

Question 14 **(7 marks)**

A small drone is launched and, after hovering in an initial position, it flies in a straight line under the control of its operator. The position of the drone from the operator is given by

$\underset{\sim}{r}(t) = \begin{pmatrix} 100+0.5t \\ 0.6t \\ 50-0.02t \end{pmatrix}$ metres, where t is the time in seconds it has been flying in a straight line.

The top of a mobile phone tower is positioned at $200\underset{\sim}{i} + 150\underset{\sim}{j} + 30\underset{\sim}{k}$ relative to the operator i.e. the mobile phone tower is 30 metres tall.



(a) After two minutes of flight, how high is the drone above the ground? (2 marks)

Solution
$\underset{\sim}{r}(120) = \begin{pmatrix} 100+0.5(120) \\ 0.6(120) \\ 50-0.02(120) \end{pmatrix} = \begin{pmatrix} 160 \\ 72 \\ 47.6 \end{pmatrix}$ $\therefore$ The drone is 47.6 metres above the ground.
Specific behaviours
✓ substitutes $t = 120$ and evaluates component(s) correctly ✓ writes a conclusion for the height of the drone (uses the third component)

Question 14 (continued)

(b) Write the expression for the position vector of the drone from the top of the phone tower after t seconds. (1 mark)

Solution
$\overrightarrow{TD}(t) = \begin{pmatrix} 100+0.5t \\ 0.6t \\ 50-0.02t \end{pmatrix} - \begin{pmatrix} 200 \\ 150 \\ 30 \end{pmatrix} = \begin{pmatrix} 0.5t-100 \\ 0.6t-150 \\ 20-0.02t \end{pmatrix}$
Specific behaviours
✓ writes a separation vector correctly (using the correct order of subtraction)

The operator knows that the drone will not strike the mobile phone tower. However, the operator does not know that the drone will cause interference when it is less than 50 metres from the top of the tower.

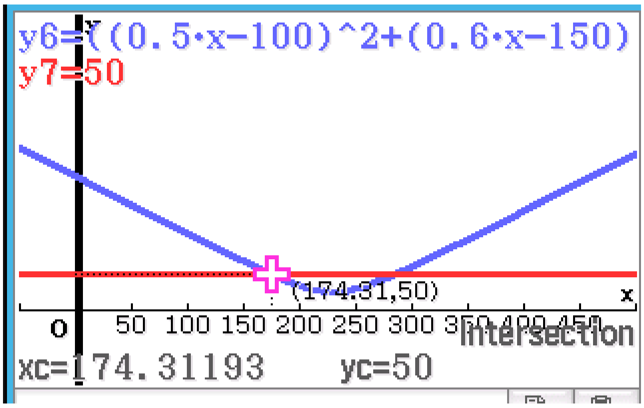
(c) Determine whether the drone will cause interference to the mobile phone tower and, if so, for how long will this occur, correct to the nearest second. (4 marks)

Solution

Require $\left|\overrightarrow{TD}(t)\right| < 50$ for interference.

$$\left|\overrightarrow{TD}(t)\right| = \sqrt{(0.5t-100)^2 + (0.6t-150)^2 + (20-0.02t)^2}$$

Using CAS: We find that when $t = 174.31$ and $t = 285.71$ to give $y = 50$



Hence YES the drone will cause interference and the period of time will be:

$\Delta t = 285.71 - 174.31 = 111.4$ seconds

i.e. the drone interferes for approx. 111 seconds i.e. 1 minute and 51 seconds.

Note: The closest approach is 24.62 metres after 230.01 seconds

Specific behaviours

✓ forms the equation or inequality that $\left|\overrightarrow{TD}(t)\right| < 50$

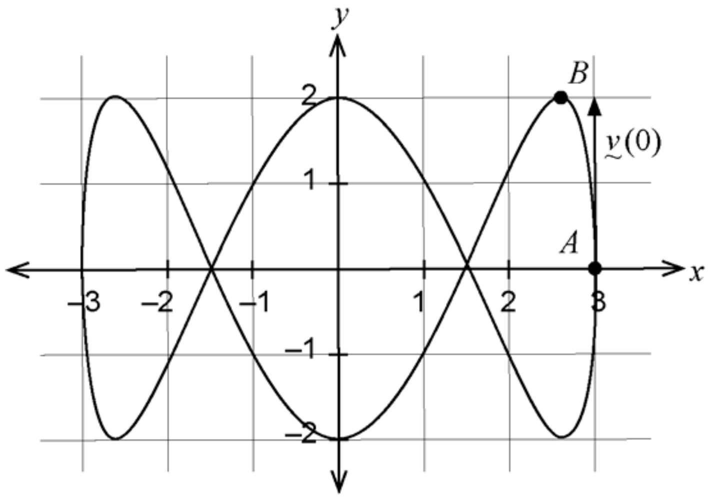
✓ forms the expression for the magnitude of the separation correctly

✓ states that the drone does interfere with the phone tower

✓ deduces how long the interference occurs to the nearest second

Question 15 **(13 marks)**

A battery-powered model race car moves around a race track as indicated in the diagram below. The car's initial position is point A.



At any time t seconds, the velocity vector $\underset{\sim}{v}(t)$ of the model race car is given by:

$$\underset{\sim}{v}(t) = \begin{pmatrix} -\sin\left(\frac{t}{3}\right) \\ 2\cos(t) \end{pmatrix}$$ metres per second.

(a) Determine the initial velocity vector and show this on the diagram above. (2 marks)

Solution
$\underset{\sim}{v}(0) = \begin{pmatrix} -\sin(0) \\ 2\cos 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ i.e. The initial velocity vector is $2\underset{\sim}{j}$ m/sec i.e. the initial velocity is 2 m/sec UPWARDS or the positive y direction.
Specific behaviours
✓ determines the vector velocity components correctly ✓ draws the vector correctly on the diagram

(b) Write an expression that will determine the change in displacement over the first $\frac{3\pi}{2}$ seconds. (2 marks)

Solution
$\Delta\underset{\sim}{r} = \int_0^{\frac{3\pi}{2}} \underset{\sim}{v}(t)\, dt$ i.e. the definite integral of the velocity vector
Specific behaviours
✓ writes a definite integral with the correct limits ✓ uses the velocity vector (with correct notation) as the integrand

Question 15 (continued)

(c) Determine the displacement vector $\underset{\sim}{r}(t)$. (3 marks)

Solution
$\underset{\sim}{r}(t) = \int \underset{\sim}{v}(t)\,dt = \int \begin{pmatrix} -\sin\left(\frac{t}{3}\right) \\ 2\cos(t) \end{pmatrix} dt = \begin{pmatrix} 3\cos\left(\frac{t}{3}\right)+c \\ 2\sin(t)+k \end{pmatrix}$ Since $\underset{\sim}{r}(0)=\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ then $c=0,\ k=0$ i.e. $\underset{\sim}{r}(t) = \begin{pmatrix} 3\cos\left(\frac{t}{3}\right) \\ 2\sin(t) \end{pmatrix}$
Specific behaviours
✓ writes the displacement vector function as the integral of the velocity vector function   ✓ anti-differentiates each component correctly   ✓ uses $\underset{\sim}{r}(0)$ to determine the constants of integration correctly

It can be shown that the model race car's speed is at a minimum when it reaches point B on the track, one of the sharpest points on the curve.

(d) Determine the acceleration vector $\underset{\sim}{a}$ when the car reaches point B, giving components correct to 0.01. (3 marks)

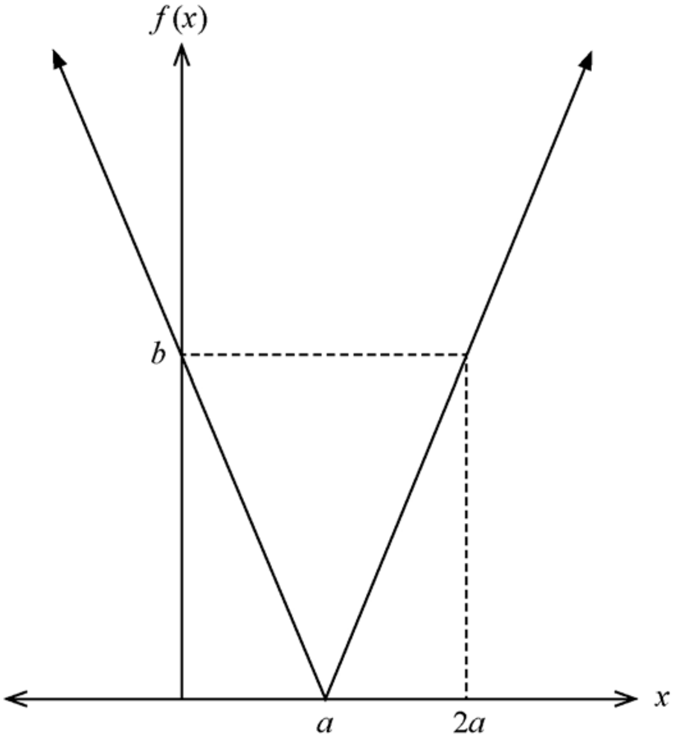
Solution
Car is at point B when $y=2$ i.e. $2\sin(t)=2$ i.e. at $t=\frac{\pi}{2}$ sec $\underset{\sim}{a}(t)=\underset{\sim}{v}'(t) = \begin{pmatrix} -\frac{1}{3}\cos\left(\frac{t}{3}\right) \\ -2\sin(t) \end{pmatrix}$ Hence $a\left(\frac{\pi}{2}\right) = \begin{pmatrix} -\frac{1}{3}\cos\left(\frac{\pi}{6}\right) \\ -2\sin\left(\frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} -0.29 \\ -2 \end{pmatrix}$ m/sec^2
Specific behaviours
✓ detemines the value of t when the car is at point B ✓ differentiates the velocity vector correctly   ✓ evaluates the components correctly (no penalty for incorrect rounding)

(e) Determine the distance, correct to 0.01 metres, that the model race car travels in completing one lap of the track. (3 marks)

Solution
One lap of the circuit is the interval $0 \le t \le 6\pi$ since $3\cos\left(\frac{t}{3}\right)=3$ for one circuit.   Distance $= \int_0^{6\pi} \left\|\underset{\sim}{v}(t)\right\| dt = \int_0^{6\pi} \sqrt{\sin^2\left(\frac{t}{3}\right)+4\cos^2(t)}\,dt$ $= 28.1645...$ i.e. the model race car travels 28.16 metres in completing one lap of the track
Specific behaviours
✓ determines the value of t when the car completes one lap   ✓ writes a definite integral using the correct expression for the speed function   ✓ evaluates correctly to 0.01 metres

Question 16 **(8 marks)**

Function *f* is defined by its graph shown below. The constants $a, b > 0$ where $b > a$.



(a) Determine the defining rule for function $f(x)$ in terms of a, b. (3 marks)

Solution
From the graph, *f* is an absolute value function of the form: $f(x) = k\lvert x-a\rvert$ Using $f(0)=b$ then $b=k\lvert 0-a\rvert$ i.e. $b=k(a)$ $\therefore$ $k=\dfrac{b}{a}$ Hence $f(x) = \dfrac{b}{a}\lvert x-a\rvert$ OR $f(x) = \left\lvert \dfrac{bx}{a}-b\right\rvert$
Specific behaviours
✓ writes an absolute value function ✓ uses the expression $\lvert x-a\rvert$ ✓ uses the vertical scale factor $\dfrac{b}{a}$

Alternative Solution
Consider f as a piecewise linear function : Slopes are $-\dfrac{b}{a}$ and $\dfrac{b}{a}$ $f(x) = \begin{cases} -\left(\dfrac{b}{a}\right)x+b, & x<a \\ \left(\dfrac{b}{a}\right)x-b, & x \ge a \end{cases}$
Specific behaviours
✓ writes the domain for each component correctly ✓ uses the gradient $\dfrac{b}{a}$ in forming linear functions ✓ writes the correct expression for each linear component

Question 16 (continued)

(b) By using the substitution $u = 2x - a$, determine an expression, in terms of a, b, for the value of $\int_{\frac{a}{2}}^{a} f(2x-a)\,dx$. (5 marks)

Solution
Using $u = 2x - a$ then for $x = \frac{a}{2}$, $u = 0$ and $x = a$, $u = a$ $\frac{du}{dx} = 2 \quad \therefore \; dx = \frac{du}{2}$ $\int_{\frac{a}{2}}^{a} f(2x-a)\,dx = \int_{0}^{a} f(u)\frac{du}{2}$ $= \frac{1}{2}\int_{0}^{a} f(u)\,du$ $= \frac{1}{2} \times$ (Area under the graph of f from $x = 0$ to $x = a$) $= \frac{1}{2} \times \left(\frac{1}{2} \times a \times b\right)$ $= \frac{ab}{4}$
Specific behaviours
✓ changes the limits of integration correctly ✓ writes dx in terms of du correctly ✓ writes the integral in terms of u correctly ✓ identifies the integral as being equal to the area under the graph from $x = 0$ to $x = a$ ✓ writes the value for the integral in terms of a, b correctly

Question 17 **(11 marks)**

After t seconds, the displacement x centimetres of a small mass attached to a spring, oscillates about a fixed point O according to the differential equation $\frac{d^2x}{dt^2} = -\pi^2 x$.

The initial velocity is 8π centimetres per second and the initial displacement is zero.

(a) Determine the function $x(t)$ that gives the displacement of the mass at time t. (3 marks)

Solution
From the differential equation $\frac{d^2x}{dt^2} = -\pi^2 x$ hence $n^2 = \pi^2$ i.e. $n = \pi$
This has the general solution $x(t) = A\sin(nt+\alpha)$ for simple harmonic motion.
i.e. $x(t) = A\sin(\pi t+\alpha)$ i.e. $x(0) = 0$ gives $0 = A\sin(\alpha)$
$\therefore$ $v(t) = A\pi\cos(\pi t+\alpha)$ i.e. $v(0) = 8\pi$ gives $8\pi = A\pi\cos(\alpha)$
Solving gives $\alpha = 0$, $A = 8$
i.e. $x(t) = 8\sin(\pi t)$
Specific behaviours
✓ uses a trigonometric function (either sine or cosine) for displacement
✓ determines the amplitude coefficient A correctly
✓ determines the phase coefficient α correctly

Alternative Solution
From the differential equation $\frac{d^2x}{dt^2} = -\pi^2 x$ hence $n^2 = \pi^2$ i.e. $n = \pi$
This has the general solution $x(t) = A\cos(nt+\alpha)$ for simple harmonic motion.
i.e. $x(t) = A\cos(\pi t+\alpha)$ i.e. $x(0) = 0$ gives $0 = A\cos(\alpha)$
$\therefore$ $v(t) = -A\pi\sin(\pi t+\alpha)$ i.e. $v(0) = 8\pi$ gives $8\pi = -A\pi\sin(\alpha)$
Solving gives $\alpha = -\frac{\pi}{2}$, $A = 8$
i.e. $x(t) = 8\cos\left(\pi t - \frac{\pi}{2}\right) = 8\cos\left(\frac{\pi}{2} - \pi t\right) = 8\sin(\pi t)$
Specific behaviours
✓ uses a trigonometric function (either sine or cosine) for displacement
✓ determines the amplitude coefficient A correctly
✓ determines the phase coefficient α correctly

Question 17 (continued)

(b) Calculate the distance the mass travels during the first 5 seconds. (3 marks)

Solution
From S.H.M. we know that the period $T = \dfrac{2\pi}{\pi} = 2$ seconds. In each period of oscillation the mass will move a distance of $4A = 32$ cm Hence over 5 seconds, Distance $= 2(4A) + 2A = 10A = 80$ cm
Specific behaviours
✓ determines the period of oscillation ✓ states that during one oscillation the distance travelled is $4A$ ✓ determine the distance travelled for 5 seconds correctly

Alternative Solution
Distance $= \int_0^5 \lvert v(t)\rvert\, dt = \int_0^5 \left\lvert 8\pi \sin\left(\pi t - \dfrac{\pi}{2}\right)\right\rvert dt$ or $\int_0^5 \lvert 8\pi \cos(\pi t)\rvert\, dt$ $= 80$ cm
Specific behaviours
✓ writes the definite integral using the absolute value of velocity ✓ uses the correct expression for the velocity function ✓ evaluates the integral correctly

The differential equation $\dfrac{d^2x}{dt^2} = -\pi^2 x$ assumes that the amplitude of oscillation A is a constant over time.

Now assume that friction reduces the amplitude of the oscillation according to the equation $\dfrac{dA}{dt} = -0.4A$. Also assume $A(0) = 8$ centimetres.

(c) Determine the function $A(t)$ that gives the amplitude of the mass. (2 marks)

Solution
From the equation $\dfrac{dA}{dt} = -0.4A$ $\quad A(t) = A(0)e^{-kt}$ is a solution $\therefore$ $A(0) = 8$ and $k = 0.4$ i.e. $A(t) = 8e^{-0.4t}$ is the function for the amplitude
Specific behaviours
✓ uses an exponential function for the amplitude, with $A(0) = 8$ ✓ determines the value for k correctly

As time passes, the amplitude continues to decrease to the point at which the small mass appears to stop oscillating. This occurs when the amplitude is less than 0.01 cm.

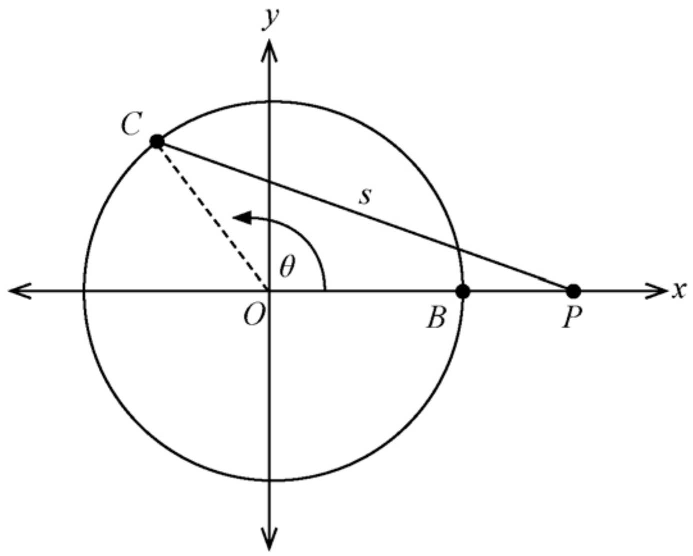
(d) Determine, correct to the nearest 0.1 seconds, how long it takes for the small mass to appear to stop oscillating. (3 marks)

Solution
We require $A(t) < 0.01$ i.e. $8e^{-0.4t} < 0.01$ Solving gives $t = 16.711...$ i.e. It will take 16.8 seconds for the small mass to appear to stop oscillating.
Specific behaviours
✓ forms an inequality (or equation) to solve for t ✓ solves the inequality (or equation) correctly ✓ concludes correctly for the value of t to 0.1 seconds

Question 18 **(10 marks)**

A young child rides on a merry-go-round at a carnival. The merry-go-round has a radius of 5 metres and completes one revolution every 12 seconds. The parent of the young child stands and watches at point P, exactly 3 metres away from point B.

The ride begins at point B, when the child is closest to the parent, and the merry-go-round rotates in an anti-clockwise direction at a constant speed. At any point in time, point C is the position of the child on the merry-go-round.



Let t = the number of seconds the ride has been in progress (from starting at point B)
$s = PC$ = the distance that the child is from the parent (metres)
θ = size of $\angle BOC$ (radians)

(a) Show that $\dfrac{d\theta}{dt} = \dfrac{\pi}{6}$ radians per second. (1 mark)

Solution
The merry go-round does one revolution of 2π radians every 12 seconds, so $\dfrac{d\theta}{dt} = \dfrac{2\pi}{12} = \dfrac{\pi}{6}$ radians per second.
Specific behaviours
✓ states that 2π radians is traversed in 12 seconds

(b) Show that $s^2 = 89 - 80\cos\theta$. (1 mark)

Solution
In ΔPOC: Applying the Cosine Rule $s^2 = 8^2 + 5^2 - 2(8)(5)\cos\theta$ i.e. $s^2 = 89 - 80\cos\theta$
Specific behaviours
✓ applies the cosine rule correctly in ΔPOC

(c) By differentiating $s^2 = 89 - 80\cos\theta$ implicitly with respect to time t, determine correct to the nearest 0.01 metre per second, the rate at which the child is moving away from the parent when the ride has been in progress for 4 seconds. (4 marks)

Solution
Require the value of $\frac{ds}{dt}$ when $t = 4$ i.e. when $\theta = \frac{4\pi}{6} = \frac{2\pi}{3}$ Differentiating $s^2 = 89 - 80\cos\theta$ implicitly with respect to time : $2s.\frac{ds}{dt} = -80(-\sin\theta).\frac{d\theta}{dt}$ when $\theta = \frac{2\pi}{3}$ $s^2 = 89 - 80\left(-\frac{1}{2}\right) = 129$ i.e. $s = \sqrt{129}$ i.e. $2\sqrt{129}\frac{ds}{dt} = 80\left(\sin\left(\frac{2\pi}{3}\right)\right)\times\frac{\pi}{6}$ i.e. $\frac{ds}{dt} = 80\left(\frac{\sqrt{3}}{2}\right)\times\frac{\pi}{6} \times \frac{1}{2\sqrt{129}} = 1.5969....$ m/sec Hence after 4 seconds, the child is moving away at a rate of 1.60 metres per second.
Specific behaviours
✓ determines the correct values for θ and s when $t = 4$ ✓✓ differentiates implicitly with respect to time correctly ✓ evaluates correctly (no penalty for incorrect rounding)

Question 18 (continued)

The parent notices that the child appears to move away from point P at varying speeds.

(d) Determine the value for $\cos\theta$ when the rate $\dfrac{ds}{dt}$ is a maximum. (4 marks)

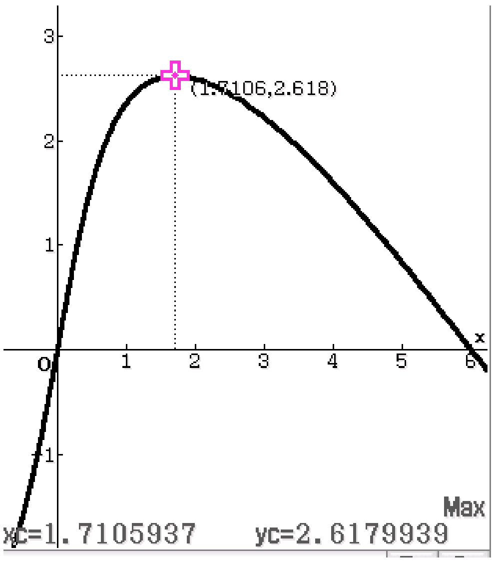
Solution
From $s = \sqrt{89-80\cos\theta} = \sqrt{89-80\cos\left(\dfrac{\pi t}{6}\right)}$ and substituting into

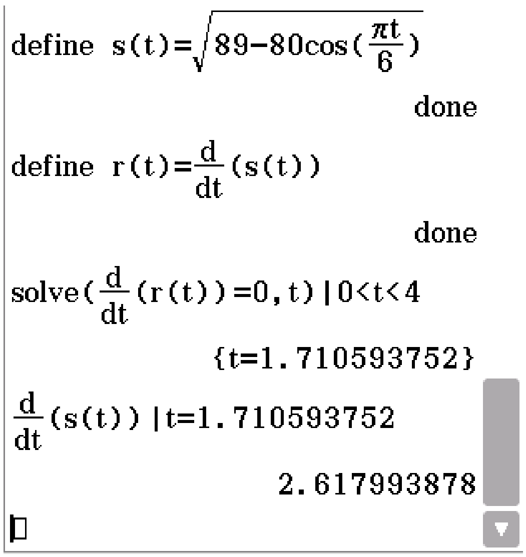
$2s.\dfrac{ds}{dt} = 80(\sin\theta).\dfrac{d\theta}{dt}$ we obtain $\dfrac{ds}{dt} = \dfrac{80\pi}{6}\sin\theta \times \dfrac{1}{2\sqrt{89-80\cos\theta}}$

i.e. $\dfrac{ds}{dt} = \dfrac{80\pi}{6}\sin\left(\dfrac{\pi t}{6}\right)\times\dfrac{1}{2\sqrt{89-80\cos\left(\dfrac{\pi t}{6}\right)}}$

Using CAS we can define $s(\theta)$ or $s(t)$:

$\therefore$ Rate $r(t)=\dfrac{ds}{dt}$





Plotting the graph of $r(t)=\dfrac{ds}{dt}$: OR Determine when $\dfrac{dr}{dt}=\dfrac{d^2s}{dt^2}=0$

There is a maximum TP at $t=1.71059$ sec Solving from CAS $t=1.71059...$ sec

Hence $\theta=\dfrac{\pi t}{6} = 0.8956...$ radians

$\therefore$ $\cos\theta = 0.625 = \dfrac{5}{8}$ for the maximum value of $\dfrac{ds}{dt}$.

Specific behaviours
✓ states the definition for the distance $s(\theta)$ or $s(t)$
✓ states that $\dfrac{d^2s}{dt^2}=0$ is the condition for maximum $\dfrac{ds}{dt}$
✓ solves for the value of θ or t correctly
✓ solves for the value of $\cos\theta$ correctly

Alternative Solution
The maximum of $\frac{ds}{dt}$ will occur when $\frac{d^2s}{dt^2}=0$ Differentiating $2s.\frac{ds}{dt} = (80\sin\theta).\frac{\pi}{6}$ implicitly with respect to time: $2s.\left(\frac{d^2s}{dt^2}\right) + 2.\left(\frac{ds}{dt}\right)\left(\frac{ds}{dt}\right) = 80\left(\frac{\pi}{6}\right)\cos\theta.\left(\frac{d\theta}{dt}\right)$ i.e. $2s.(0) + 2\left(\frac{80\pi\sin\theta}{6\times 2s}\right)^2 = 80\left(\frac{\pi}{6}\right)\cos\theta.\left(\frac{\pi}{6}\right)$ ……. substituting $\frac{d^2s}{dt^2}=0$ i.e. $2\times\frac{80^2\pi^2\sin^2\theta}{6^2\times 4(89-80\cos\theta)} = \frac{80\pi^2}{6^2}\cos\theta$ ……. substituting for $\frac{ds}{dt}$ and s^2 i.e. $40\sin^2\theta = (89-80\cos\theta)\cos\theta$ i.e. $40(1-\cos^2\theta) = (89-80\cos\theta)\cos\theta$ i.e. $40\cos^2\theta - 89\cos\theta + 40 = 0$ i.e. $(8\cos\theta-5)(5\cos\theta-8)=0$ $\therefore \cos\theta = \frac{5}{8}$ since $\cos\theta \neq \frac{8}{5} > 1$ Hence $\frac{ds}{dt}$ is a maximum when the child is at a position such that $\cos\theta = \frac{5}{8}$ i.e. $\theta = 0.8956...$ i.e. when $t = 1.710...$ seconds. Note : when $\cos\theta = \frac{5}{8}$, this means that $\overleftrightarrow{PC}$ is a tangent to the circle. The maximum value for $\frac{ds}{dt} = \frac{5\pi}{6} = 2.6179...$ m/sec
Specific behaviours
✓ states that $\frac{d^2s}{dt^2}=0$ is the condition for maximum $\frac{ds}{dt}$ ✓✓ differentiates implicitly again with respect to time correctly ✓ solves for $\cos\theta$ or θ or t correctly to give the position for the child

Question 19 **(7 marks)**

Consider the complex equation $2z^6 = 1+\sqrt{3}i$.

(a) Solve the above equation, giving solutions in polar form $rcis\theta$ where $0<\theta<\frac{\pi}{2}$.

(4 marks)

Solution
$z^6 = \frac{1}{2}+\frac{\sqrt{3}}{2}i = \text{cis}\left(\frac{\pi}{3}\right)$ Solutions are : $z = cis\left(\frac{\frac{\pi}{3}+2\pi k}{6}\right) = cis\left(\frac{\pi}{18}+\frac{\pi}{3}k\right)$ where $k=0,1$ i.e. $z_0 = cis\left(\frac{\pi}{18}\right) = cis\left(10^o\right)$, $z_1 = cis\left(\frac{\pi}{18}+\frac{6\pi}{18}\right) = cis\left(\frac{7\pi}{18}\right) = cis\left(70^o\right)$
Specific behaviours
✓ expresses z^6 in polar form correctly ✓ forms the correct expression for the roots using De Moivre's Theorem ✓ states that $z = cis\left(\frac{\pi}{18}\right)$ is a solution ✓ states that $z = cis\left(\frac{7\pi}{18}\right)$ is a solution

Now consider the equation $2z^n = 1+\sqrt{3}i$, where n is a positive integer.

(b) If $2z^n = 1+\sqrt{3}i$ has roots so that there are exactly 3 roots (and only 3) that lie within the first quadrant of the complex plane, determine the possible value(s) of n. Justify your answer. (3 marks)

Solution
The first solution is $z = cis\left(\frac{\pi}{3n}\right)$ is always in the first quadrant irrespective of n. There are n equally spaced roots, separated by $\frac{2\pi}{n}$. The 3rd root is in quadrant 1 so this means that: $\frac{\pi}{3n} + \left(2\times\frac{2\pi}{n}\right) < \frac{\pi}{2}$ since the argument must be less than $\frac{\pi}{2}$ $\frac{\pi}{3n} + \frac{4\pi}{n} < \frac{\pi}{2}$ $\therefore\ 2\pi + 24\pi < 3n\pi$ $\therefore\ n > \frac{26}{3}$ *i.e.* $n \geq 9$ The 4th root must be in quadrant 2 so this means that: $\frac{\pi}{3n} + \left(3\times\frac{2\pi}{n}\right) > \frac{\pi}{2}$ since the argument must be greater than $\frac{\pi}{2}$ $\frac{\pi}{3n} + \frac{6\pi}{n} > \frac{\pi}{2}$ $\therefore\ 2\pi + 36\pi > 3n\pi$ $\therefore\ n < \frac{38}{3}$ *i.e.* $n \leq 12$ Hence it must be true that $n = 9, 10, 11,$ or 12. Alternatively: There must be either 2 or 3 solutions within each of the other 3 quadrants. i.e. $n = 3+2+2+2 = 9$ or $n = 3+3+2+2 = 10$ or $n = 3+3+3+2 = 11$ or $n = 3+3+3+3 = 12$ Hence $n = 9, 10, 11$ or 12.
Specific behaviours
✓ states that $n = 12$ is a possibility ✓ states that $n = 9, 10, 11$ are the other possibilities ✓ justifies why there are 4 possibilities

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